CS103
Winter 2025

Lecture 08:

Set Theory Revisited

Outline for Today

- ***Proofs on Sets***
  - Making our intuitions rigorous.
- ***Formal Set Definitions***
  - What do our terms mean?
- ***Appendices: Examples***
  - Sample proofs to help you get the hang of the ideas here.

Recap from Last Time

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

New Stuff!

Proving Results from Set Theory

Claim: If A, B, and C are sets where $A \in B$ and $B \in C$, then $A \in C$.

Proof (?): Assume A, B, and C are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that A is contained in B. Since $B \in C$, we know that B is contained in C. Therefore, because A is contained in B and B is contained in C, we know that A is contained is C. This means that $A \in C$, as required. ■

Claim: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof (?): Assume A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that A is contained in B. Since $B \subseteq C$, we know that B is contained in C. Therefore, because A is contained in B and B is contained in C, we know that A is contained is C. This means that $A \subseteq C$, as required. ■

Which (if any) of these claims are true?
Answer at

https://cs103.stanford.edu/pollev

Lie: If A, B, and C are sets where $A \in B$ and $B \in C$, then $A \in C$.

Bad Proof: Assume A, B, and C are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that A is contained in B. Since $B \in C$, we know that B is contained in C. Therefore, because A is contained in B and B is contained in C, we know that A is contained is C. This means that $A \in C$, as required. ■

Theorem: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Bad Proof: Assume A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that A is contained in B. Since $B \subseteq C$, we know that B is contained in C. Therefore, because A is contained in B and B is contained in C, we know that A is contained is C. This means that $A \subseteq C$, as required. ■

This can't be a good proof;
the same basic argument
proves a false claim!

Claim: Let A, B, and C be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

Proof (?): Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of A are in B. Since $A \subseteq C$, all elements of A are also in C. Therefore, all elements of A are in both B and C. Therefore, we see that $A \subseteq B \cap C$. ■

Claim: Let A, B, and C be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

Proof (?): Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of A are in B and there are other elements of B. Since $A \subsetneq C$, all elements of A are also in C and there are other elements of C. Therefore, all elements of A are in both B and C, and there are some other elements in B and C. Therefore, we see that $A \subsetneq B \cap C$. ■

(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)

Which (if any) of these claims are true?
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Theorem: Let A, B, and C be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

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(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)

What Went Wrong?

- The style of arguments you've just seen are ***not*** how to prove results on sets.
- As you've seen:
  - The reliance on high-level terms like "contained" is not mathematically precise.
  - A discussion of "all elements" of a set is not how to reason about collections of objects.
- ***Question:*** How do we write rigorous proofs about sets?

The Importance of Definitions

- As you saw last week, formal definitions underpin mathematical proofs.
- The major issue from the previous proofs is that we haven't defined what our terms mean.
  - How do we define what $A \in B$ means?
  - How do we define what $A \subseteq B$ means?
  - How do we define what $A \cap B$ means?
- Think back to our proof triangle: we currently have intuitions for these concepts, but not formal definitions.

	Is defined as…	If you ***assume*** this is true…	To ***prove*** that this is true…
$S \subseteq T$			
$S = T$			
$x \in S \cap T$			
$x \in S \cup T$			
$X \in \wp(S)$			
$x \in \{ y \mid P(y) \}$			

Proofs on Subsets

Theorem: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Defining Subsets

- Formally speaking, if S and T are sets, we say that $S \subseteq T$ when the following holds:

$$\forall x \in S.\ x \in T$$

- Now, suppose you're working with a proof where you encounter $S \subseteq T$. Think back to the proof table.
  - To ***assume*** that $S \subseteq T$, what should you do?
  - To ***prove*** that $S \subseteq T$, what should you do?

Answer at

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A Correct Proof on Sets

Theorem: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

Theorem: If A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

Bad Proof: Assume A, B, and C are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that A is contained in B. Since $B \subseteq C$, we know that B is contained in C. Therefore, because A is contained in B and B is contained in C, we know that A is contained is C. This means that $A \subseteq C$, as required. ■

Unions and Intersections

Theorem: Let A, B, and C be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Unions and Intersections

- The statement $x \in S \cap T$ is defined as

$$\boldsymbol{x \in S \quad \wedge \quad x \in T.}$$

- The statement $x \in S \cup T$ is defined as

$$\boldsymbol{x \in S \quad \vee \quad x \in T.}$$

- These are operational definitions: they show how unions and intersections interact with the $\in$ relation rather than saying what the union or intersection of two sets "are."

Theorem: Let A, B, and C be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Proof: Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required. ■

Theorem: Let A, B, and C be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Proof: Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required. ■

Bad Proof: Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of A are in B. Since $A \subseteq C$, all elements of A are also in C. Therefore, all elements of A are in both B and C. Therefore, we see that $A \subseteq B \cap C$. ■

Set Equality

Theorem: Let A and B be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

Proof: See appendix!

Set-Builder Notation

Set-Builder Notation

- Let S be the set defined here:

$$S = \{ \ n \mid n \in \mathbb{N} \text{ and } n \geq 137 \ \}$$

- Now imagine you have some quantity x. Based on this…
  - … if you ***assume*** that $x \in S$, what does that tell you about x?
  - … if you need to ***prove*** that $x \in S$, what do you need to prove?

Answer at

https://cs103.stanford.edu/pollev

Set-Builder Notation

- Like unions and intersections, we have an operational definition for set-builder notation. It's the following:

Let $S = \{ y \mid P(y) \}$.
Then $x \in S$ when $P(x)$ is true.

- So, for example:
  - $x \in \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$ means $x \in \mathbb{N}$ and x is even.
  - $x \in \{ n \mid \exists k \in \mathbb{N}.\ n = 2k + 1\}$ means that there is a $k \in \mathbb{N}$ where $x = 2k + 1$. (Equivalently, x is an odd natural number)
- ***Key Point:*** The placeholder variable disappears in all these examples. After all, *it's just a placeholder.*

Proofs on Set-Builder Notation

Some Useful Notation

- If n is a natural number, we define the set **[*n*]** as follows:

$$[n] = \{ \ k \mid k \in \mathbb{N} \ \land \ k < n \ \}$$

- So, for example:
  - $[3] = \{0, 1, 2\}$
  - $[0] = \emptyset$
  - $[5] = \{0, 1, 2, 3, 4\}$

Theorem: If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

Theorem: If m, $n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

What We're Assuming	*What We Need to Prove*
$m \in \mathbb{N}$	$\cancel{[m] \subseteq [n]}$
$n \in \mathbb{N}$	$\cancel{\forall x \in [m].\ x \in [n]}$
$m < n$	$x \in \mathbb{N}$
$[z] = \{\ k \mid k \in \mathbb{N} \land k < z\ \}$	$x < n$
$x \in [m]$	
$x \in \mathbb{N}$	
$x < m$	

Theorem: If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

Proof: Assume m and n are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$. Then, because $x < m$ and $m < n$, we know that $x < n$. Collectively this means that $x \in \mathbb{N}$ and $x < n$, so $x \in [n]$, as required. ■

Notice that *there is no set-builder notation in this proof*. We were able to avoid it by using the rules for what $x \in \{ y \mid P(y) \}$ say to do.

	Is defined as...	If you ***assume*** this is true...	To ***prove*** that this is true...
$S \subseteq T$	$\forall x \in S.\ x \in T$	Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$.	Ask the reader to pick an $x \in S$. Then prove $x \in T$
$S = T$	$S \subseteq T \land T \subseteq S$	Assume $S \subseteq T$ and $T \subseteq S$.	Prove $S \subseteq T$. Also prove $T \subseteq S$.
$x \in S \cap T$	$x \in S \land x \in T$	Assume $x \in S$. Then assume $x \in T$.	Prove $x \in S$. Also prove $x \in T$.
$x \in S \cup T$	$x \in S \lor x \in T$	Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$.	Either prove $x \in S$ or prove $x \in T$.
$X \in \wp(S)$	$X \subseteq S$.	Assume $X \subseteq S$.	Prove $X \subseteq S$.
$x \in \{ y \mid P(y) \}$	$P(x)$	Assume $P(x)$.	Prove $P(x)$.

Announcements

- ***Upcoming Deadlines***
  - (**Today**, 11:59 PM): Left-handed exam desk requests
  - (**Friday**, 1:00 PM): Problem Set 3
    - Start early and make slow and steady progress.
    - Attend OH sooner rather than later.
  - (**Friday**, 11:59 PM): Attendance opt-out
  - (**Sunday**, 7 PM): Midterm 1 Bonus

Announcements

- ***Hodgepodge***
  - Problem Set 3 LaTeX template updated Sunday evening.
  - Task map updated.
  - New “Resources” for the week now tagged as such in drop-down on course website.
  - Looking for a pset partner? Post on Ed!
  - Sean’s OH have moved to CoDa.
- ***Midterm 1***
  - Logistics page unlocked.
  - So many practice problems.
  - More on Wednesday.

Readings

- ***Read "Guide to Proofs on Discrete Structures."***
  - There's additional guidance and practice on the assume/prove dichotomy and how it manifests in problem-solving.
- ***Read "Discrete Structures Proofwriting Checklist."***
  - Keep the items here in mind when writing proofs. We'll use this when grading your problem set.
- ***Read "Guide to Proofs on Sets."***
  - There's some good worked examples in there to supplement today's lecture, several of which will be relevant for the problem set.

Next Time

- ***Graph Theory***
  - A ubiquitous, powerful abstraction with applications throughout computer science.
- ***Vertex Covers***
  - Making sure tourists don’t get lost.
- ***Independent Sets***
  - Helping the recovery of the California Condor.

Appendix: More Sample Set Proofs

Theorem: Let A, B, C, and D be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

What I'm Assuming	*What I Need to Show*
$A \subseteq C$	$\cancel{A \cup B \subseteq C \cup D}$
$\forall z \in A.\ z \in C$	$\cancel{\forall x \in A \cup B.\ x \in C \cup D}$
$B \subseteq D$	$x \in C$ or $x \in D$
$\forall z \in B.\ z \in D$	
$x \in A \cup B$	
Case 1: $x \in A$	
Case 2: $x \in B$	

Theorem: Let A, B, C, and D be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

Proof: Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

Case 1: $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

Case 2: $x \in B$. Then because $B \subseteq D$ and $x \in B$ we have $x \in D$, so $x \in C \cup D$.

In either case, we see that $x \in C \cup D$, as required. ■

Theorem: Let A and B be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

Proof: We will prove each direction of implication.

(⇒) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

(⇐) Assume $A \cup B = B$. We need to show that $A \subseteq B$. So pick an $x \in A$; we need to show that $x \in B$.

Since $x \in A$, we know that $x \in A \cup B$. And since $x \in A \cup B$ and $A \cup B = B$, we see that $x \in B$, as required. ■

Theorem: Let A and B be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

What I'm Assuming		*What I Need to Show*
$\wp(A) = \wp(B)$		$\cancel{A = B}$
$\wp(A) \subseteq \wp(B)$		$\cancel{A \subseteq B}$
$\forall Z \in \wp(A).\ Z \in \wp(B)$		$\cancel{\forall x \in A.}\ x \in B.$
$\wp(B) \subseteq \wp(A)$		$\cancel{B \subseteq A}$
$\forall Z \in \wp(B).\ Z \in \wp(A)$		$\forall z \in B.\ z \in A.$
$x \in A$	$x \in B$	
$\{x\} \subseteq A$	$\{x\} \subseteq B$	
$\{x\} \in \wp(A)$	$\{x\} \in \wp(B)$	

Theorem: Let A and B be sets. If $\wp(A) = \wp(B)$, then $A = B$.

Proof: Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of A and B are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$. This means that $\{x\} \in \wp(A)$, and since $\wp(A) \subseteq \wp(B)$ we know $\{x\} \in \wp(B)$. Thus we see that $\{x\} \subseteq B$, which in turn means that $x \in B$, as required. ■